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Evolution of the Quantitative Measures of Quantum Information: on the Path to a Unified Theory of Quantum Information

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Abstract—The most general approaches to determining the quantitative measure of quantum information are analyzed based on physical content. Dynamic information is proposed as an alternative to coherent information. In a sense, the former corresponds to the use of a set of all possible states in the Hilbert space of a quantum system as the information available for free exchange.

1. INTRODUCTION

Comprehension of the information content in quantum information physics somewhat differs from that in classical physics. In classical analysis, the information is primarily related to the communication problems, whereas quantum physics predominantly concentrates on its nonclassical features. This leads to a certain gap between the classical and quantum theories. We expect that this gap can be bridged based on the fundamental principles of quantum theory, since they include the principles of classical physics.

In the study of this problem, there exist two opposite tendencies related to different interpretations of the relationship between quantum mechanics and information theory (see, for example, recent works [1, 2]).

One of the approaches suggests that the principles of quantum mechanics are modified in accordance with the existing principles of quantum information theory [1]. In the framework of another approach, the information theory is built on standard axioms of quantum mechanics with allowance for their corresponding interpretation and modification [2]. In this work, we employ the latter approach, since, in our opinion, the existing information theory is not a closed theory. The key point in the information theory is the method of quantitative measurements. Therefore, we concentrate on determining the quantitative measures of quantum information and characterizing their most general properties.

2. FUNDAMENTAL ROLE OF THE INCOMPATIBILITY OF QUANTUM STATES

To start the analysis, let us discuss the fundamental difference between quantum and classical systems. In spite of the multitude of specific quantum effects, there is a single reason behind them, lying in the incompatibility of quantum events (it is basically impossible to describe them in the framework of classical logic). The
mathematic manifestation is the nonorthogonality of 1
the wave functions that describe the quantum events
related to the noncommuting physical quantities. The 2
first consequence of the nonorthogonality is the quantum uncertainty in the total ensemble of quantum events described by the Hilbert space of the quantum system. This leads to the probability of entanglement of the given quantum state with another state even in the case when the observer has direct access to the quantum system. For example, Fig. 1a demonstrates the probability of entanglement of the true state $|\alpha\rangle$ with an arbitrary state $|\beta\rangle$ at the Bloch sphere: only the orthogonal state cannot be entangled with the original one.

Therefore, there exists an ultimate amount of information distinguishable in the total ensemble of states. For a two-dimensional space (qubit), this amount is about 0.279 bit. Even in the case of infinite-dimensional space, the limit is 0.61 bit owing to the mutual masking of the states related to their quantum uncertainty.

Another important manifestation of the quantum uncertainty is seen in the statistical relations between two identical quantum systems (in the case under consideration, these are two two-level atoms in the same energy state). Let us consider only two eigenstates and reduce the multitude of the quantum states to two two-point sets with purely classical probabilistic properties. Then, in the states $|1\rangle|1\rangle$ and $|2\rangle|2\rangle$, each atom is an equivalent copy of the other. Taking into account the nonorthogonality of the internal states of the atoms, we arrive at the quantum uncertainty inside the atoms. In addition, the atomic quantum fluctuations are independent, and the nonorthogonal states cannot mimic each other. If the state of the first atom is classified incorrectly, the error for the other atom is different (Fig. 1b) and the identity of all the states is impossible.

This corresponds to the uniqueness of a quantum system and the impossibility of equivalent copying of

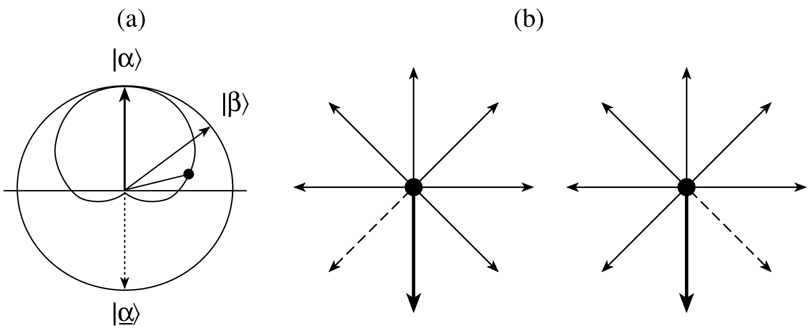

Fig. 1. (a) The uncertainty of the quantum states $|\beta\rangle$ of physical quantities for the exactly known state $|\alpha\rangle$ of the quantum system. (b) Nonequivalence of two two-level atoms in one and the same state.

all its states by another system. The operator of the corresponding rms deviation of states is written as

$$\hat{\varepsilon} = \int (|\alpha\rangle\langle\alpha| \otimes \hat{I}_B - \hat{I}_A \otimes |\alpha\rangle\langle\alpha|)^2 \frac{dV_\alpha}{D}$$

$$= \|0\rangle\rangle\langle\langle 0\| + \frac{1}{3}\sum_{k=1}^{3} \|k\rangle\rangle\langle\langle k\| \geqq \frac{1}{3}.$$

It is seen that it is always positive for any joint state $\hat{\rho}_{AB}$ of the atoms. Thus, states $|\alpha\rangle$ in system A cannot be reproduced simultaneously in the corresponding 3 states $|\beta\rangle$ of system B. They can only be teleported (reproduced and automatically destroyed in the original system). A trivial example of teleportation is the dynamic transformation in a closed system, when the original state disappears and a new state is created at another time moment.

The entanglement of quantum systems is the most fundamental manifestation of the quantum incompatibility being the basis of applications for quantum information. This results from the exchange of the incompatible internal states of quantum systems taking place in the presence of the dynamic interaction. If the statistical coupling between the subsystems caused by the interaction is not used and the properties of only one system are analyzed, the entanglement leads only to classical fluctuations (mixed character of the one-particle states $\hat{\rho}_A$ and $\hat{\rho}_B$). Using the quantum information from both systems, one can reveal the nonclassical properties of the information employed in quantum cryptography, quantum computing, etc.

3. EXISTING MEASURES OF QUANTUM INFORMATION

The basic notion of information theory is a communication channel. It describes the interaction between two systems which can be named input and output. Based on the type of compatibility of the input and output states, one can give the most general classification of the existing quantitative information measures.

The Shannon information is determined via the joint probability distribution of the input and output of the classical channel, when the channel itself is set by the transient probability distribution. The amount of information is expressed in terms of the Shannon entropies of both systems and each system taken separately. It can be represented in two equivalent forms: asymmetric (the difference between the total output entropy and the mean value of the conditional output entropy) and symmetric (the difference between the sum of the output and input entropies and the joint entropy). The main result of the Shannon information theory is the possibility of error-free transfer of this amount of information upon multiple use of the channel. The Shannon information is interpreted as a unified information measure independent of the physical nature of the channel.

A semiclassical channel differs from a classical one in the quantum character of the input. One can easily determine the corresponding semiclassical quantitative measure, since the input–output compatibility makes it possible to simply substitute the input-dependent density matrix of the quantum channel for the transient distribution in the Shannon formula. This definition is nontrivial in view of the fact that the given amount of information can be realized in the form of classical information stable against copying and any applications.

The purpose of introducing coherent information is to describe purely quantum properties of information in a completely quantum channel. Such a channel is described using the density matrix $\hat{\rho}_A$ of the input A related to the output B by a transformation $\mathcal{N}$ that, in the general case, takes into account both the reversible dynamics and the noises of a reservoir (environment E). A qualitatively new content is the mutual incompatibility of the input and output. However, the interpretation of the physical meaning of coherent information makes it necessary to introduce a reference system R compatible with the output. This system supplements the input system, and, jointly, they are described by a pure state with the wave function $|\Psi_{AR}\rangle$. The corresponding one-particle input density matrix is written as $\hat{\rho}_A$ =

$\mathrm{Tr}_B|\Psi_{AR}\rangle\langle\Psi_{AR}|$. Owing to such an interpretation of the mixed input ensemble, the content of the original quantum information is the entanglement of the input and the reference system. Note that the transformation of state does not affect the reference system, which appears to be compatible with the output as well. Thus, coherent information can be introduced as the difference between the output entropy and the joint entropy of the output and the reference system. The coherent information related to the flow of the quantum incompatible states from the input to the output characterizes the states of one quantum system at two different time moments.

However, a special analysis is needed to determine the possibility of freely manipulating the states of the system corresponding to two different time moments in the case of mathematical interpretation of the information. Note also the nonpositiveness of the coherent information, which makes it difficult to analyze it as a variant of the Shannon information.

The definition of the compatible information from [3] fully agrees with the principles of the Shannon information theory. The compatible information is introduced as the amount of classical information contained in a completely quantum channel at given input and output ensembles of quantum states set by the corresponding nonorthogonal expansions of unity or the positive operator measures (POM) at the input and output. These POM are usually associated with the quantum measurement procedure. However, in the absence of real measurements, they can be considered intrinsic characteristics of quantum ensembles observable upon the corresponding quantum measurement procedure. The most important particular cases of POM are the sets of all quantum states and the orthogonal bases in the corresponding Hilbert spaces.

We perform calculations of the coherent information for the following physical systems: a decaying two-level atom in the presence of a strong laser field, information exchange between open subsystems in the hydrogen atom, two interacting atoms in the absence of relaxation, atomic emission in free space, two atoms with radiative interaction in the Dicke problem [4], and radiative exchange of a Λ system with a vacuum field [5]. These calculations can help in reaching a more in-depth understanding of the coherent information. In particular, it can be stated that the term coherent information in its original meaning contradicts both the interpretation of information in the Shannon theory and the alternative definition of coherent information (see below). The reason for this is the impossibility of freely using the corresponding information describing systems that do not exist simultaneously. Therefore, a preferred variant is the conserved entanglement [6, 7].

The most significant qualitative features of the compatible information are as follows: (i) operation invari-
4 ance in the case of completely unselected information
4 [8], (ii) equivalence of the completely unselected information and the completely selected information averaged with respect to all orientations of the orthogonal bases describing the selected information, (iii) equiva-
lence of the maximum of unselected information and 4
distinguishable information [9], and (iv) sensitivity of the amount of compatible information to the degree of POM selectivity.

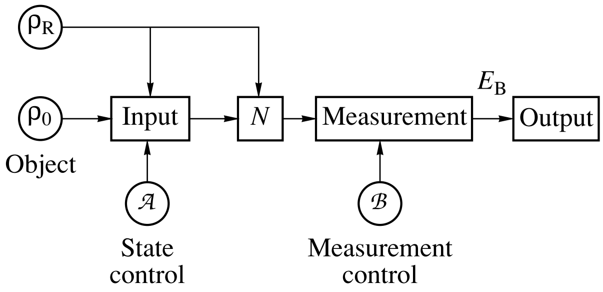

Fig. 2. General scheme for the measurements of the physical parameters of a quantum system.

An extreme example demonstrating the importance of the selection of quantum ensembles involved in the input–output coupling is given by two sets of N input and output systems. In the absence of the selection of states, the maximum amount of information in these systems is given by the corresponding distinguishable information, which tends to 0.61 bit at large N. If we select the states representing direct products of the arbitrary one-particle states, the accessible information is proportional to N and can be infinitely large.

In our opinion, the compatibility of the input and output information is the key point in the information optimization of the experimental scheme from [7] for measuring the parameters of a quantum system (Fig. 2).

The generalized characteristic of a quantum system
is set by two superoperator measures $\mathscr{A}(da)$ and $\mathscr{B}(db)$ 5
that determine the action upon the quantum object and the measurement procedure at the output of the information channel. Finally, these measures make it possible to set the statistical relation between the parameters measured and the results of the measurements and to calculate the corresponding amount of Shannon information that must be optimized. A specific model example of the above optimization is the measurement of the unknown dipole moment of a two-level atom excited by a laser field and detection of the photons emitted.

4. RECONSTRUCTION OF QUANTUM INFORMATION THEORY BASED ON THE PRIMARY PRINCIPLES OF QUANTUM MECHANICS

Based on the above discussion and an analysis of the results of physical applications, it can be concluded that the notion of coherent information is qualitatively incompatible with the Shannon theory. Note that this statement is not associated with the known arguments from [10] regarding the inadequacy of the Shannon theory for the purposes of quantum information theory related to the dependence of the amount of information on the measurement procedure. Here, we concentrate on the problem of simultaneously using the states of the quantum input and output that do not exist simultaneously in reality.

The problem of introducing the corresponding quantitative measure of the quantum information can be solved in the framework of the Shannon information theory provided that the incompatible ensembles of quantum states allow adequate introduction of the corresponding classical ensembles. Then, all the types of information are based on a unified Shannon information measure and their sum total can be considered as a realization of a unified theory. Note that such a theory should shed light on the information content of the primary principles of quantum mechanics.

A possible variant of formulating the basic qualitative principles of such a theory would involve the following assumptions.

(i) Quantum mechanics does not need an external observer; the information exchange is contained in the reversible quantum dynamics.

(ii) The foundation of the quantum information theory must be a unified measure based on the Shannon information. This measure must be valid for all physical types of quantum information exchange.

Let us discuss the first postulate as applied to the mathematical description of the time evolution of quantum systems A, B, C, … . If we restrict consideration to the Schrödinger representation, the states of all physical systems are mutually compatible and the incompatibility effects lead only to the entanglement of their joint wave function. The primary physical communication channels can be related only to compatible ensembles of quantum systems, which finally make it possible to express such fundamental notions as physical spacetime. The relations between the states at various time moments (if they appear) must be reduced to relations between mutually compatible ensembles with only internal incompatibility.

Is it possible to equivalently represent a quantum-incompatible ensemble by a classical ensemble? The answer is positive but in a limited sense. All the states of the Hilbert space can be one-to-one related to the orthogonal set of states in the extended Hilbert space with infinite dimensionality. This yields a statistical distribution for the nonorthogonal set of all states in the original space that can adequately reproduce all the quantum probabilities.

Such a representation makes it possible to simultaneously describe (in terms of the corresponding uncertainty of the classical ensemble) the mutual input–output incompatibility and to introduce the corresponding quantitative measure for the information exchanged. Here, we restrict ourselves to the analysis of the quantum transformation of a noisy channel that, in the general case, represents a nonlinear one-to-one mapping $|\alpha\rangle \longrightarrow |\beta(\alpha, \xi)\rangle$ of the input quantum state dependent on the classical noise ξ. This representation is widely used in the computer simulation of open quantum systems employing the so-called quantum trajectories. It is analyzed in both purely mathematical and physical works. Starting from a fixed value of the noise variable, we arrive at the one-to-one mapping in the ensemble of entangled states mapping the original ensemble in a compatible form. Finally, we obtain the corresponding classical probability distributions for the quantum channel under consideration. It contains all the output quantum information regarding the original state of the input ensemble, which disappears after this transformation.

The corresponding amount of Shannon information can be defined as the dynamic information characterizing the information relations between the result of the dynamic transformation of the quantum system and its possible original states. In contrast to coherent information, it describes information represented as a classical (orthogonal) set of states $\|\alpha\rangle\rangle$ selected from the total quantum set of states in the Hilbert space $H_A \otimes H_a$ of the system under consideration with the Hilbert space H_A and an auxiliary system with the infinite-dimensional Hilbert space H_a. They characterize the original quantum states $|\alpha\rangle \in H_A$ with allowance for their quantum uncertainty adequately represented by the fluctuations in H_a.

5. CONCLUSIONS

Thus, a single non-Shannon-type information measure (coherent information) has a Shannon-type analog representing dynamic information.

This work is devoted to the search for ways to find a unified theory, whose basic principles can be found in different fields. Nevertheless, the search must be continued, since only such a quantum information theory can be merged with physics.

The results of this work were reported at the Klyshko memorial seminar. In this connection, we draw attention to the substantial influence of the works of this prominent scientist in the field of quantum optics and of his scientific legacy in general in the development of quantum theory, including quantum information theory.

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SPELL: 1. mathematic, 2. noncommuting, 3. teleported, 4. unselected, 5. superoperator