Coherent-information analysis of quantum channels in simple quantum systems

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The coherent-information concept is used to analyze a variety of simple quantum systems. Coherent information was calculated for the information decay in a two-level atom in the presence of an external resonant field, for the information exchange between two coupled two-level atoms, and for the information transfer from a two-level atom to another atom and to a photon field. The coherent information is shown to be equal to zero for all full-measurement procedures, but it completely retains its original value for quantum duplication. Transmission of information from one open subsystem to another one in the entire closed system is analyzed to learn quantum information about the forbidden atomic transition via a dipole active transition of the same atom. It is argued that coherent information can be used effectively to quantify the information channels in physical systems where quantum coherence plays an important role.

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I. INTRODUCTION

The concept of a noisy quantum channel may be used in many information-carrying applications, such as quantum communication, quantum cryptography, and quantum computers [1]. Shannon's theory of information [2–5] is a purely classical one and cannot be applied to quantum mechanical systems. Therefore, much recent work has been done on quantum analogs of the Shannon theory [6–11]. The *coherent information* introduced in [7,9] is suggested to be analogous to the concept of *mutual information* in classical information theory. It is defined by

$$I_c = S_{\mathrm{out}} - S_e, \tag{1}$$

where S_{out} is the entropy of the information channel output and S_e is the *entropy exchange* [6,9] taken from the channel reservoir. If $S_{\mathrm{out}} - S_e > 0$, then, expressed in *qubits*, I_c describes a binary logarithm of the Hilbert space dimension, all states of which are transmitted with the probability $p=1$ in the limit of infinitely large ergodic ensembles. Otherwise, we set $I_c = 0$.

The validity of the coherent information concept was proved in [9,10], and it was used successfully for quantifying the resources needed to perform physical tasks. Coherent information is expected to be as universal as its classical analog, Shannon information, and it characterizes a quantum information channel regardless of the nature of both quantum information and quantum noise. In contrast to Shannon information in classical physics, however, coherent information is expected to play a more essential role in quantum physics. The capacity of information channels in classical physics can be estimated, in most cases, even without relying on any information theory, at least within an order of magnitude. This, however, is not feasible in quantum physics and the coherent information concept, or a similar concept, must be used to quantify the information capacity of the channel. An analysis of the quantum information potentially available in physical systems is especially important for planning experiments in new fields of physics, such as quantum computations, quantum communications, and quantum cryptography [1,11], where the coherent information of the quantum channel determines its potential efficiency.

In this paper, we apply the coherent-information concept to an analysis of the quantum information exchange between two systems, which in general may have essentially different Hilbert spaces. For this purpose, we must specify the noisy quantum information channel and its corresponding superoperator $\mathcal{S}$, which transforms the initial state of the first system into the final state of another system. A classification scheme for possible quantum channels connecting two quantum systems is shown in Fig. 1 [12]. In addition to the two-time channels shown in the figure, we consider also their one-time analogs. Two-time quantum channels are widely used in quantum communications and measurements, whereas one-time quantum channels are appropriate for quantum comput-

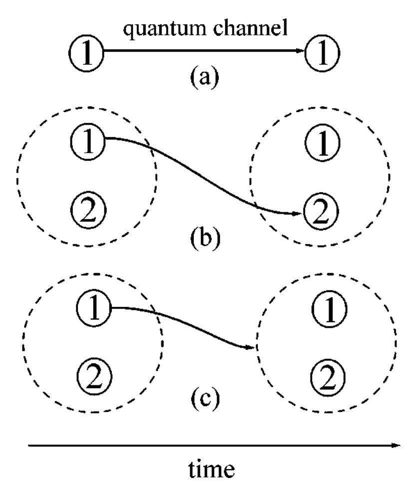

FIG. 1. Classification of possible quantum channels connecting two quantum systems. $1 \rightarrow 1$, information is transmitted from the initial state of the system to its final state (a); $1 \rightarrow 2$, information is transmitted from subsystem 1 of the system $(1+2)$ to subsystem 2 of the system (b); $1 \rightarrow (1+2)$, information is transmitted from subsystem 1 of the system $(1+2)$ to the whole system $(1+2)$ (c).

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ing and quantum teleportation.

The paper is organized as follows. In Sec. II we explain key definitions and review the superoperator representation technique, which is used throughout the paper. In the following sections we consider a variety of quantum channels that correspond to the classification scheme shown in Fig. 1. Sec. III discusses the coherent-information transfer between quantum states of a two-level atom (TLA) in a resonant laser field at two time instants [Fig. 1(a)]. The same type of quantum channel $(1\rightarrow 1)$ can be considered for a system that contains two (or more) subsystems. This case is analyzed in Sec. IV using a spinless model of the hydrogen atom as an example. Coherent-information transfer between two different quantum systems is considered in Sec. V. The analysis includes coherent-information transfer between (i) two unitary coupled TLAs [Fig. 1(b)], (ii) two TLAs coupled via the measuring procedure [Fig. 1(b)], (iii) an arbitrary system and its duplication [Fig. 1(c)], (iv) a TLA in the free-space photon field [Fig. 1(b)], and (v) two TLAs via the free-space photon field [Fig. 1(b)]. Finally, Sec. VI concludes with a summary of our results.

II. KEY DEFINITIONS AND CALCULATION TECHNIQUE

A. Notations and superoperator representation technique

This section explains key notations and briefly reviews the symbolic superoperator representation technique [13], which is especially convenient for the mathematical treatment of coherent-information transmission through a noisy quantum channel. The most general symbolic representation of a superoperator is defined by the expression

$$\mathcal{S}=\sum \hat{s}_{kl}\langle k|\odot|l\rangle, \tag{2}$$

where the substitution symbol $\odot$ must be replaced by the transforming operator variable and $\langle k|$ is an arbitrarily chosen vector basis in Hilbert space H, to which the transformed operators are applied. In order to correctly apply this transformation to a density matrix, operators $\hat{s}_{kl}$ must obey the positivity condition for the block operator $\hat{S}=(\hat{s}_{kl})$ [14] and orthonormalization condition

$$\mathrm{Tr}\,\hat{s}_{kl}=\delta_{kl}, \tag{3}$$

which provides normalization for all normalized operators $\hat{\rho}$ with $\mathrm{Tr}\,\hat{\rho}=1$.

Using symbolic representation (2), one can easily represent the production of superoperators $\mathcal{S}_1,\mathcal{S}_2$, which constitutes a symbolic representation of the superoperator algebra. For $\hat{s}_{kl}=|k\rangle\langle l|$ Eq. (2) results in the identity superoperator $\mathcal{I}$, and for $\hat{s}_{kl}=|k\rangle\langle k|\delta_{kl}$—in the quantum reduction superoperator $\mathcal{R}=\Sigma|k\rangle\langle k|\odot|k\rangle\langle k|$. The case of $\hat{s}_{kl}=\delta_{kl}$ represents the trace superoperator $\mathrm{Tr}\odot$, which is a linear functional in the density-matrix space. The correspondence between the matrix representation $S=(S_{mn})$ of the superoperator $\mathcal{S}$ in orthonormalized operator basis $\hat{e}_k$ and its symbolic representation (2) is given by

$$\hat{s}_{kl}=\mathcal{S}(|k\rangle\langle l|)=\sum_{mn} S_{mn}\langle l|\hat{e}_n|k\rangle\hat{e}_m \tag{4}$$

and can be easily checked by substituting it in Eq. (2) and comparing with the standard definition of matrix elements $\mathcal{S}\hat{e}_n=\Sigma_m S_{mn}\hat{e}_m$.

B. The calculation of coherent information

The entropy exchange in Eq. (1) for the coherent information is defined as

$$S_e=S(\hat{\rho}_\alpha),\quad S(\hat{\rho})=-\mathrm{Tr}\,\hat{\rho}\log_2\hat{\rho}, \tag{5}$$

where the joint input-output density matrix $\hat{\rho}_\alpha$ is given, in accordance with [9,15], by

$$\hat{\rho}_\alpha=\sum_{ij} \mathcal{S}(|\rho_i\rangle\langle\rho_j|)\otimes|\overline{\rho_i}\rangle\langle\overline{\rho_j}|. \tag{6}$$

Here $|\rho_i\rangle=\hat{\rho}_{\mathrm{in}}^{1/4}|i\rangle$ are the transformed eigenvectors of the input density matrix $\hat{\rho}_{\mathrm{in}}=\Sigma p_i|i\rangle\langle i|$; the bar symbol stands for complex conjugation and $\mathcal{S}$ is the channel input-output superoperator, so that the output density matrix $\hat{\rho}_{\mathrm{out}}=\mathcal{S}\hat{\rho}_{\mathrm{in}}$. Using superoperator representation (2) within the above defined eigenbasis $|i\rangle$, the density matrix (6) takes the form

$$\hat{\rho}_\alpha=\sum_{ij} (p_ip_j)^{1/4}\hat{s}_{ij}\otimes|\overline{\rho_i}\rangle\langle\overline{\rho_j}|, \tag{7}$$

where operators $\hat{s}_{ij}$ represent the states of the output. Both the input and output marginal density matrices are given by the trace over the corresponding complementary system: $\hat{\rho}_{\mathrm{out}}=\mathrm{Tr}_{\mathrm{in}}\hat{\rho}_\alpha$, $\overline{\hat{\rho}_{\mathrm{in}}}=\mathrm{Tr}_{\mathrm{out}}\hat{\rho}_\alpha$. Finally, the coherent information (1) can be calculated, keeping in mind that $S_{\mathrm{out}}=S(\hat{\rho}_{\mathrm{out}})$.

1. Two-time coherent information for two quantum systems

For the coherent information transfer between two quantum systems through the quantum channels shown in Figs. 1(b) and 1(c) $[1\rightarrow 2$ or $1\rightarrow(1+2)]$, the initial joint density matrix must be taken in the product form $\hat{\rho}_{1+2}=\hat{\rho}_{\mathrm{in}}\otimes\hat{\rho}_2$, where $\hat{\rho}_{\mathrm{in}}=\hat{\rho}_1$ and $\hat{\rho}_2$ are the initial marginal density matrices, the first one being an input. For the $1\rightarrow 2$ quantum channel, the output is the state of the second system, since a transformation on these two systems is made and a certain amount of information is transmitted into the second system from the initial state of the first one.

The dynamical evolution of the joint $(1+2)$ system is given by a superoperator $\mathcal{S}_{1+2}$ and the corresponding channel transformation superoperator, which converts $\hat{\rho}_{\mathrm{out}}=\mathcal{S}\hat{\rho}_{\mathrm{in}}$, can be written as

$$\mathcal{S}=\mathrm{Tr}_1\mathcal{S}_{1+2}(\odot\otimes\hat{\rho}_2),$$

where the trace is taken over the final state of the first system. The transformation is described in terms of Eq. (2) for the joint system as

$$\mathcal{S}=\sum_{k\kappa l\lambda}\sum_{n}\langle n|\hat{s}_{k\kappa,l\lambda}|n\rangle\langle\kappa|\hat{\rho}_2|\lambda\rangle\langle k|\odot|l\rangle, \tag{8}$$

where the product basis $|k\rangle|\kappa\rangle$ is used and indexes k, κ stand for the first and second quantum systems, respectively. The operator coefficients $\hat{s}_{kl}$ in Eq. (2) now take the form

$$\hat{s}_{kl}=\sum_{\kappa\lambda}\sum_{n}\langle n|\hat{s}_{k\kappa,l\lambda}|n\rangle\langle\kappa|\hat{\rho}_2|\lambda\rangle. \tag{9}$$

Superoperator $\mathcal{S}$ depends on both the dynamical transformation $\mathcal{S}_{1+2}$ and the initial state $\hat{\rho}_2$, and couples the initial state of the first system with the final state of the second system.

2. One-time coherent information

One-time information quantities can be easily calculated if the corresponding joint density matrix is known. In the case of a single system, the corresponding channel is described by the identity superoperator $\mathcal{I}$. For the joint input-output density matrix (6), we get a pure state $\hat{\rho}_\alpha=\Sigma_i|\rho_i\rangle|\rho_i\rangle\Sigma_j\langle\rho_j|\langle\rho_j|$ and then calculate the entropy exchange $S_e=0$ and the coherent information $I_c=S_{\text{out}}=S_{\text{in}}$. In the case of two systems, the input-output density matrix is the joint density matrix $\hat{\rho}_{1+2}$, and the corresponding coherent information in system 2 on system 1 at time t is $I_c(t)=S[\hat{\rho}_2(t)]-S[\hat{\rho}_{1+2}(t)]$. In the case of unitary dynamics and a pure initial state of the second system, all initial eigenstates $|i\rangle$ of the first system transform into the corresponding orthogonal set $\Psi_i(t)$ of the $(1+2)$ system, so that the joint entropy is time independent and the coherent information yields $I_c(t)=S[\hat{\rho}_2(t)]-S[\hat{\rho}_1(0)]$. If the initial state of the first system is also a pure state, we get simply $I_c(t)=S[\hat{\rho}_2(t)]$. For the TLA case, this simply yields $I_c=1$ qubit, if a maximally entangled state of two-atom qubits is achieved.

III. TLA IN A RESONANT LASER FIELD

In this section we discuss the coherent-information transfer between the quantum states of a TLA in a resonant laser field at two time instants [Fig. 1(a)]. Such a quantum channel with pure dephasing in the absence of an external field was considered in [9]. In a more general case, coherent information, based on the joint input-output density matrix (6), can be readily calculated by using the matrix representation technique for the relaxation dynamics superoperator. An interesting question is how the coherent information depends on the applied resonance field.

The field changes the relaxation rates of the TLA. These rates are presented with the real parts of the eigenvalues λ_k of the dynamical Liouvillian $\mathcal{L}=\mathcal{L}_r+\mathcal{L}_E$ of the TLA, where $\mathcal{L}_r$ and $\mathcal{L}_E$ stand for the relaxation and field interaction Liouville superoperators. For simplicity, we will consider here relaxation caused only by pure dephasing, combined with the laser field interaction. The corresponding Liouvillian matrix in the basis of $\hat{e}_k=\{\hat{I},\hat{\sigma}_3,\hat{\sigma}_1,\hat{\sigma}_2\}$ reads

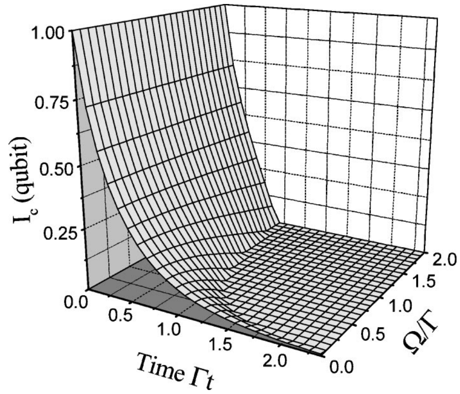

FIG. 2. The coherent information transmitted between the states of the TLA at two time instants, $t=0$ and $t>0$, versus time Γt and the Rabi frequency Ω/Γ (both are dimensionless).

$$L=\begin{pmatrix}0&0&0&0\\0&0&0&\Omega\\0&0&-\Gamma&0\\0&-\Omega&0&-\Gamma\end{pmatrix}, \tag{10}$$

where Γ is the pure dephasing rate, Ω is the Rabi frequency, and $\hat{\sigma}_1,\hat{\sigma}_2,\hat{\sigma}_3$ are the Pauli matrices. The eigenvalues of the matrix (10) can be readily calculated and are given by

$$\lambda_k=\{0,-\Gamma,-(\Gamma+\sqrt{\Gamma^2-4\Omega^2})/2,-(\Gamma-\sqrt{\Gamma^2-4\Omega^2})/2\}.$$

These values are affected by the resonant laser field with respect to the unperturbed values $0,\Gamma$, which also affects the resonant fluorescence spectrum of the TLA. At $\Omega>\Gamma/2$ it results in the so-called Mollow-triplet structure, centered at the transition frequency, which has been predicted theoretically [16] and subsequently confirmed experimentally [17].

From the information point of view, the resonant laser field might reduce the coherent-information decay rate and, therefore, lead to the increase of information, although this information gain could intuitively be expected only from the laser-induced reduction of the relaxation rates of the relaxation superoperator $\mathcal{L}_r$ itself [18–21].

Calculating the matrix of the evolution superoperator $\mathcal{S}=\exp(\mathcal{L}t)$ and using its corresponding representation (2), the joint density matrix may be calculated analytically (6). Then [with the help of Eqs (5) and (1)], the coherent information left in the TLAs state at time t about its initial state may be calculated. The initial state is chosen in the form of the maximum entropy density matrix $\hat{\rho}_0=\hat{I}/2$. The results of our calculations are presented in Fig. 2. They show the typical threshold-type dependence of the coherent information versus time, which is determined by the loss of coherence in the system. Also, the coherent information does not increase with an increase of the laser field intensity, as might be expected. The coherent information even decreases as the Rabi frequency increases.

In addition, the results show a singularity in the first derivative of the coherent-information dependence at time $t=0$, which is a characteristic feature of the starting point of the decay of coherent quantum information. Initially, the input-output density matrix (6) of the TLA is a pure state $\hat{\rho}_\alpha=\Psi\Psi^+$ with the input-output wave function $\Psi=\Sigma\sqrt{p_i}|i\rangle|i\rangle$. Its eigenvalues λ_k and the probabilities of the corresponding eigenstates are all equal to zero, except for the eigenstate corresponding to Ψ. Due to the singularity of the entropy function $-\Sigma\lambda_k\log\lambda_k$ at $\lambda_k=0$ the derivative of the corresponding exchange entropy also shows a logarithmic singularity.

Another interesting feature of coherent information is its dependence on the initial (input) state $\hat{\rho}_{in}$. If it were possible, $\hat{\rho}_{in}$ might be chosen in the form of the eigenoperator

$$\hat{\rho}_{\rm in}=\sum_{l=1}^{4}|k_{\rm min}\rangle_l\hat{e}_l$$

of the Liouville superoperators, where $|k_{\rm min}\rangle$ is the eigenvector corresponding to the minimum value $|{\rm Re}\lambda_k|>0$ of the matrix L. Yet the vector $|k_{\rm min}\rangle$ is equal to $\{0,(\Gamma+\sqrt{\Gamma^2-4\Omega^2})/2\Omega,0,1\}$, which corresponds to the linear space of operators with zero trace due to the zero value of the first component. Therefore, the coherent-information decay rate cannot be reduced by reducing the corresponding decay of atomic coherence.

IV. COHERENT-INFORMATION TRANSFER BETWEEN TWO SUBSYSTEMS OF THE SAME QUANTUM SYSTEM

In this section we investigate the quantum channel $[1\to1$, Fig. 1(a)$]$ between two open subsystems A and B of a closed system $A+B$ having a common Hilbert space $sp(H_A,H_B)$, where H_A and H_B are the Hilbert subspaces of the subsystems A and B, respectively.

In classical information theory, this situation corresponds to the transmission of part $A\subset X$ of the values of an input random variable $x\in X$. The situation where a receiver receives no message is also informative and means that x belongs to the supplement of A, $x\in\overline{A}$. It can be described by the *choice* transformation $\mathcal{C}=P_A+P_0(1-P_A)$, where P_A is the projection operator from X onto the subset A, $P_Ax=x$ for $x\in A$ and $P_Ax=\emptyset$ (empty set) for $x\in\overline{A}$, P_0 is the projection from X onto an independent single-point set X_0, and $P_0x=X_0$. This transformation corresponds to the classical *reduction* channel, resulting in information loss only if $\overline{A}$ is not a single point. If $\overline{A}$ is a single point, we are able to get a maximum of one bit of information, for $\overline{A}$ can provide another point of the bit, so that for an input bit we have no loss of information.

In quantum mechanics, the corresponding reduction channel is represented as the choice superoperator

$$\mathcal{C}=\hat{P}_A\odot\hat{P}_A+|0\rangle\langle0|{\rm Tr}(1-\hat{P}_A)\odot(1-\hat{P}_A), \tag{11}$$

where state $|0\rangle$ is a quantum analog of the classical single-point set, which is separate from all other states. Equation (11) defines a positive and trace-preserving transformation, which can appropriately describe the coherent-information transfer between subsets of the entire system. The last term in Eq. (11) represents the total norm preservation, if all the states outside the output B set are included. In our case, these states are included in the incoherent $|0\rangle\langle0|$ form, which in contrast to the classical one-bit analog of a TLA yields no coherent information due to the complete destruction of the coherence.

Considering the coherent information transmitted from part A to part B of the system, which evolves in time, we deal with the channel superoperator

$$\mathcal{S}_{AB}=\mathcal{C}_B\mathcal{S}_0(t)\mathcal{C}_A,\quad \mathcal{S}_0(t)=U(t)\odot U^{-1}(t) \tag{12}$$

with $U(t)$ being the time evolution unitary operator. Here the input choice superoperator $\mathcal{C}_A$ is shown just to define the total channel superoperator, regardless of the input density matrix. Otherwise, $\mathcal{C}_A$ is already accounted for in the input density matrix $\hat{\rho}_{\rm in}$, defined as the operator in the corresponding subspace H_A of the total Hilbert space H.

Let us assume that the dynamical evolution of the system is determined and the Bohr frequencies ω_k and the corresponding eigenstates $|k\rangle$ are found. Then, representing the projectors in terms of the corresponding input $|\psi_l\rangle$ and output $|\varphi_m\rangle$ wave functions, Eq. (12) gives the specified time evolution form

$$\mathcal{S}_{AB}(t)=\sum_{ll'\in A}\left[\hat{s}_{ll'}(t)+|0\rangle\langle0|\sum_{m\notin B}\langle\varphi_m|\psi_l(t)\rangle\langle\psi_{l'}(t)|\varphi_m\rangle\right]\times\langle\psi_l|\odot|\psi_{l'}\rangle,$$

$$\hat{s}_{ll'}(t)=\sum_{mm'\in B}\langle\varphi_m|\psi_l(t)\rangle\langle\psi_{l'}(t)|\varphi_{m'}\rangle|\varphi_m\rangle\langle\varphi_{m'}|,$$

$$|\psi_l(t)\rangle=\sum_k e^{-i\omega_kt}\langle k|\psi_l\rangle|k\rangle. \tag{13}$$

Let us consider the case of the orthogonal subsets of input/output wave functions, which is of special interest. Then, if there is only one common state $|\phi\rangle$ in the sets $|\psi_l\rangle$, $|\varphi_m\rangle$ and $U(t_0)=1$ holds for some t_0, we get

$$\mathcal{S}_{AB}(t_0)=|\phi\rangle\langle\phi|\odot|\phi\rangle\langle\phi|+|0\rangle\langle0|\sum_{\varphi_m\neq\phi}\langle\varphi_m|\odot|\varphi_m\rangle,$$

which means that the quantum system is reduced into a classical bit of the states $|\phi\rangle$ and $|0\rangle$ and no coherent information is stored in the subsystem B. Nevertheless, if the eigenstates $|k\rangle$ of $U(t)$ do not coincide with the input/output states $|\psi_l\rangle$, $|\varphi_m\rangle$ the coherent information will increase with the time evolution. Hence, the information capacity of the channel is determined by quantum coupling of the input and output.

To illustrate the coherent-information transfer through the quantum channel considered in this section, let us analyze a typical intra-atomic channel between two two-level systems formed of two pairs of orthogonal states $A=\{|\psi_0\rangle,|\psi_1\rangle\}$ and $B=\{|\psi_0\rangle,|\psi_2\rangle\}$ of the same atom. A spinless model of the

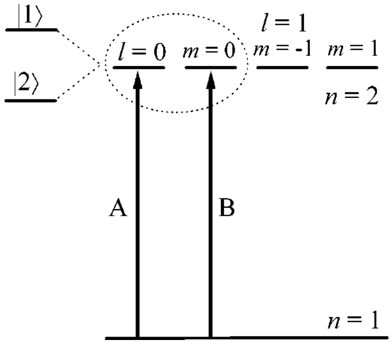

FIG. 3. A spinless model of the hydrogen atom. The information channel is made of the input forbidden $nlm \rightarrow n'l'm'$ transition 100–200 and the output dipole active 100–210 transition.

hydrogen atom could serve as such a system (Fig. 3): ψ_0 is the ground s state with $n=1$, $\psi_{1,2}$ is the s state with $l=0, m=0$ and the p-state with $l=1, m=0$ of the first excited state with $n=2$, respectively.

In the absence of an external field, this quantum channel transmits no coherent information, as the $l=0, m=0$ and $l=1, m=0$ states are uncoupled. In the presence of an external electric field applied along the Z axis, the considered two out of four initially degenerated states with $n=2$ are split, due to the Stark shift into the new eigenstates $|1\rangle=(|\psi_1\rangle+|\psi_2\rangle)/\sqrt{2}, |2\rangle=(|\psi_1\rangle-|\psi_2\rangle)/\sqrt{2}$. The input $l=0$ state oscillates with the Stark shift frequency: $|\psi_1(t)\rangle=\cos(\omega_s t)|\psi_1\rangle+\sin(\omega_s t)|\psi_2\rangle$. Therefore, due to the applied electric field, the input state becomes coupled to the output state, which carries the coherent information.

For our model, Eq. (13) presents the $\hat{s}_{kl}$ operators in the form of a 3×3 matrix, where the third column and row introduce the phantom ''vacuum'' state $|0\rangle$,

$$\hat{s}_{11}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{s}_{12}=\begin{pmatrix} 0 & \sin\omega_s t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{s}_{21}=\begin{pmatrix} 0 & 0 & 0 \\ \sin\omega_s t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{s}_{22}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\omega_s t & 0 \\ 0 & 0 & \cos^2\omega_s t \end{pmatrix}.$$

Zero values of $\hat{s}_{12}, \hat{s}_{21}$ correspond to the absence of coherent information at $t=0$ or to the absence of coupling. Choosing the input matrix in the maximum entropy form $\hat{\rho}_{\rm in}=\hat{I}/2$, we get the corresponding joint input-output matrix in the form

$$\hat{\rho}_\alpha=\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{x}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{x}{2} & 0 & 0 & \frac{x^2}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-x^2}{2} \end{pmatrix},$$

where $x=\sin\omega_s t$ and the output density matrix $\hat{\rho}_{\rm out}$ is diagonal with the diagonal elements 1/2, $x^2/2$, and $(1-x^2)/2$.

Calculating nonzero eigenvalues $(1\pm x^2)/2$ of $\hat{\rho}_\alpha$ and the entropies $S_{\rm out}$, S_α, we get the coherent information

$$I_c=[(1+x^2)\log_2(1+x^2)-x^2\log_2(x^2)]/2.$$

This function is positive except for $x=0$, where the coherent information is equal to zero, and its maximum is equal to 1 qubit at $x=\pm 1$, e.g., for the precession angle $\omega_s t=\pm\pi/2$. Thus coherent information on the state of the forbidden transition is available, in principle, from a dipole transition via Stark coupling. Its time-averaged value is $\langle I_c\rangle=0.46$ qubit.

This forbidden transition was discussed in [22,23] as a potential source of information on spatial symmetry breaking caused by the weak neutral current [24,25]. For example, if $I_c=0$, only the incoherent impact of the forbidden transition (by means of the ground-state population n_0) remains and provides a classical-type of information on the interactions that cannot be observed directly. In this case, only one parameter—population—can be potentially measured, while exact knowledge of the phase of the transition demands $I_c=1$.

V. COHERENT-INFORMATION TRANSFER BETWEEN TWO QUANTUM SYSTEMS

In recent years, a few results have been published related to coherent-information transfer in a system of two TLAs, including discussion of the problem from the entanglement measure viewpoint [26] and the ''eavesdropping problem'' [27]. A number of different experiments have been proposed to study controlled entanglement between two atoms [28,29]. From the informational point of view, the coherent-information transmitted in the system of two TLAs connected by a quantum channel depends both on the specific quantum channel transformation and the initial states of the TLAs. For the latter, it seems reasonable to assume that they can be represented by the product of the independent states of each TLA: $\hat{\rho}_{1+2}=\hat{\rho}_{\rm in}\otimes\hat{\rho}_2$.

In this section we present a systematic treatment of the coherent-information transfer between two different quantum systems. The analysis includes coherent-information transfer between (i) two unitary coupled TLAs (Sec. V A), (ii) two TLAs coupled via the measuring procedure (Sec. V B), (iii) an arbitrary system and its duplicate (Sec. V C), (iv) a TLA and the free-space photon field (Sec. V D), and (v) two TLAs coupled via the free-space photon field (Sec. V E).

A. Two unitary coupled TLAs

Let us first examine a deterministic noiseless quantum channel connecting two TLAs [Fig. 1(b)]. Such a channel can be described by the unitary two-TLA transformation, which is defined by the matrix elements $U_{ki,k'i'}$ with $k,i,k',i'=1,2$. Then, the channel transformation superoperator $\mathcal{S}$ describing the transformation $\hat{\rho}_{\rm in}\rightarrow\hat{\rho}_{\rm out}=\hat{\rho}_2'$ can be written in terms of the substitution symbol [see Eq. (2)], with

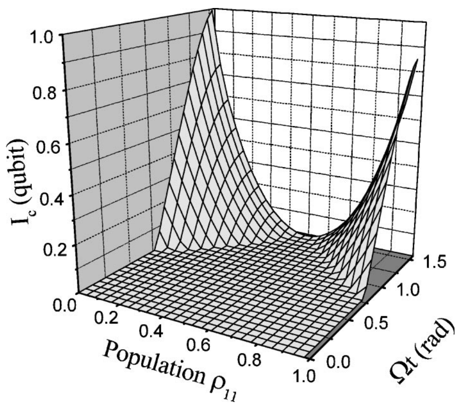

FIG. 4. The coherent information transmitted between two unitary coupled TLAs versus population ρ_{11} of the diagonal initial density matrix of the second TLA and the coupling precession angle Ωt.

operators $\hat{s}_{kl}=\Sigma_{\mu\nu}S_{kl,\mu\nu}|\mu\rangle\langle\nu|$, represented by the matrix elements of $\mathcal{S}$ [in accordance with Eqs (4) and (9)], in the following form:

$$S_{kl,\mu\nu}=\sum_{m\alpha\beta}\rho_{2\alpha\beta}U_{m\mu,k\alpha}U^{*}_{m\nu,l\beta}. \qquad (14)$$

The relation $\mathrm{Tr}\,\hat{s}_{kl}=\Sigma_{\mu}S_{kl,\mu\mu}=\delta_{kl}$ is valid here and ensures the correct normalization condition, whereas the positivity of the block matrix

$$(\hat{s}_{kl})=\begin{pmatrix}\hat{s}_{11} & \hat{s}_{12}\\ \hat{s}_{21} & \hat{s}_{22}\end{pmatrix}$$

ensures the positivity of $\mathcal{S}$.

For the no-entanglement transformation $U=U_1\otimes U_2$, Eqs (2) and (14) yield $\mathcal{S}=\hat{\rho}_2'\mathrm{Tr}\odot$, which means that the initial state $\hat{\rho}_1$ of the first TLA transfers into the final state, which is not entangled with the state $\hat{\rho}_2'=U_2\hat{\rho}_2U_2^{\dagger}$ of the second TLA.

We can simplify Eq. (14) by considering a pure state $\hat{\rho}_2$, so that together with an arbitrary choice of no-entanglement transformation U it seems reasonable to consider a special case of the pure state: $\rho_{2\alpha\beta}=\delta_{\alpha\beta}\delta_{\alpha\alpha_0}$. Also keeping in mind that $S_{kl,\mu\nu}$ is linear on the density matrix $\hat{\rho}_2$ and the coherent information I_c is a convex function of $\mathcal{S}$ [10], Eq. (14) simplifies to

$$S_{kl,\mu\nu}=\sum_{m}U_{m\mu,k\alpha_0}U^{*}_{m\nu,l\alpha_0}, \qquad (15)$$

which means that the quantum channel is described only by the unitary transformation U. Here the summation is taken over only the states $|m\rangle$ of the first TLA after the coupling transformation.

The coherent information transmitted in systems of two unitary coupled TLAs with $\hat{\rho}_{\mathrm{in}}=\hat{I}/2$ and $(\hat{\rho}_2)_{12}=\sqrt{(\hat{\rho}_2)_{11}[1-(\hat{\rho}_2)_{11}]}$ is shown in Fig. 4. A convex function of $\hat{\rho}_2$ is shown, which has the maximum on the border, $\rho_{11}=(\hat{\rho}_2)_{11}=0,1$. As in the case of a single TLA, the behavior of the coherent information preserves the typical threshold-type dependency on the coupling angle, which determines the degree of the coherent coupling of two TLAs with respect to the independent fluctuations of the second TLA.

B. Two TLAs coupled via the measuring procedure

Here we will discuss a specific type of quantum channel connecting two TLAs [30], where the superoperator $\mathcal{S}$ is defined by the *measuring procedure*, which implements a different approach to the quantum information [31] called *measured information*.

We start with a channel formed of two identical two-level systems. In terms of wave function, the corresponding *full measurement* transformation of the first TLA state is defined as

$$\psi\otimes\varphi\rightarrow\sum a_i|\phi_i\rangle|\phi_i\rangle, \quad a_i=\langle\phi_i|\psi\rangle. \qquad (16)$$

This transformation provides full entanglement of some basis states $|\phi_i\rangle$, which do not depend on the initial state φ of the second TLA. The latter serves as a measuring device, yet fully preserves information on the basis states of the first system state $\psi=\Sigma a_i|\phi_i\rangle$. Eq. (16), being a deterministic transformation of the wave function, is neither a linear nor unitary transformation with respect to φ and, therefore, cannot represent a true deterministic transformation. The corresponding representation in terms of the two-TLA density matrices has the form

$$\hat{\rho}_{12}\rightarrow\sum_i\sum_j\langle\phi_i|\langle\phi_j|\hat{\rho}_{12}|\phi_j\rangle|\phi_i\rangle|\phi_i\rangle|\phi_i\rangle\langle\phi_i|\langle\phi_i|. \qquad (17)$$

This representation is linear on $\hat{\rho}_{12}$ and satisfies the standard conditions of physical feasibility [10,32], i.e., completely positive and trace preserving. This matrix is in the form of $\Sigma p_i|\phi_i\rangle|\phi_i\rangle\langle\phi_i|\langle\phi_i|$, so that $S(\hat{\rho}_{12})=S(\hat{\rho}_2)$. Due to the classical nature of the information represented here only with the classical indexes i and in accordance with the equations of Sec. II, the single-instant coherent information is zero.

In the case of a two-time channel, the superoperator for the quantum channel connecting two TLAs can be readily derived from Eq. (2) with $\hat{s}_{kl}=|\phi_k\rangle\langle\phi_k|\delta_{kl}$, $\langle k|\rightarrow\langle\phi_k|$, and $|k\rangle\rightarrow|\phi_k\rangle$. After calculating the trace over the first TLA and replacing $\hat{\rho}_{12}$ with the substitution symbol $\odot$, the equation takes the form

$$\mathcal{M}=\sum_k\hat{P}_k\mathrm{Tr}_1\hat{e}_k\odot. \qquad (18)$$

Here $\hat{P}_k=|\phi_k\rangle\langle\phi_k|$ are the orthogonal projectors representing the eigenstates of the ''pointer'' variable of the second TLA and $\hat{E}_k=|\phi_k\rangle\langle\phi_k|$ is the orthogonal expansion of the unit (orthogonal map) formed of the same projectors. This orthogonal map determines here the quantum-to-classical reduction transformation $\mathrm{Tr}_1\hat{E}_k\odot=\langle\phi_k|\odot|\phi_k\rangle$, which represents the procedure of getting classical information k from the first system. Applying the transformation (18) to $\hat{\rho}_{\mathrm{in}}$ and using Eq. (6) for the respective output and input-output density matrices, we get

$$\hat{\rho}_{\mathrm{out}}=\sum_k \tilde{p}_k|\phi_k\rangle\langle\phi_k|,\quad \hat{\rho}_\alpha=\sum_k \tilde{p}_k|\phi_k\rangle|\pi_k\rangle\langle\pi_k|\langle\phi_k|, \tag{19}$$

where $\tilde{p}_k=\langle\phi_k|\hat{\rho}_{\mathrm{in}}|\phi_k\rangle=\Sigma_i p_i|\langle\phi_k|i\rangle|^2$ are the eigenvalues of the reduced density matrix and $|\pi_k\rangle=\Sigma_i\sqrt{p_i/\tilde{p}_k}\langle\phi_k|i\rangle|\overline{i}\rangle$ are the normalized modified input states coupled with the output states $|\phi_k\rangle$ after the measurement procedure. It is important to note [as it follows from Eq. (19)] that there is no coherent information in the system because vectors $|\phi_k\rangle$ are orthogonal and therefore the entropies of the density matrices (19) are obviously the same. Conversely, the measured information introduced in [31] is not equal to zero in this case.

We can easily generalize our result for a more general case of the quantum channel, when the second system has a different structure from the first and, therefore, they occupy different Hilbert spaces. This difference leads to the replacement of the basis states $|\phi_i\rangle$ of the second system in our previous results with another orthogonal set $|\varphi_i\rangle=V|\phi_i\rangle$, where V is an isometric transformation from the Hilbert state H_1 of the first system to the different Hilbert space H_2 of the second system. After simple straightforward calculations, the final result is the same—there is no coherent information transmitted through the quantum channel. This result is a natural feature of coherent information, in contrast to other information approaches (see, for example, Ref. [31]).

It is interesting to discuss more general measuring-type transformations, for instance, the indirect (generalized) measurement procedure. This procedure was first applied to the problems of optimal quantum detection and measurement in [33] and then, in a form of nonorthogonal expansion of unit $\hat{\mathcal{E}}(d\lambda)$, in [34] [$\hat{\mathcal{E}}(d\lambda)$ is equivalent to the positive operator-valued measure (POVM), used in the semiclassical version of quantum information and measurement theory [11,35,36]]. This indirect measuring transformation results from averaging a direct measuring transformation applied, not to the system of interest, but to its combination with an auxiliary independent system. The indirect-measurement superoperator in the general form can be written as

$$\mathcal{M}=\sum_q \hat{P}_q\mathrm{Tr}\,\hat{\mathcal{E}}_q\odot, \tag{20}$$

where $\hat{P}_q$ are the arbitrary orthogonal projectors and $\hat{\mathcal{E}}_q$ is the general-type nonorthogonal expansion of the unit in H space (POVM). Note that $\hat{\mathcal{E}}_q=|\varphi_q\rangle\langle\varphi_q|$ is a specific ''pure'' type of POVM, first used in quantum detection and estimation theory [33]. The latter describes the full measurement in $H\otimes H_a$ for the singular choice of the initial auxiliary system density matrix $\rho^a_{bc}=\delta_{b0}\delta_{bc}$.

The information transfer from the initial density matrix to the final output state is represented in Eq. (20) via the coupling provided by indexes q. Because the number N_q of q values can be greater than $\mathrm{Dim}\,H$, it seems reasonable to suggest that some output coherent information is left about the input state. The corresponding output and input-output density matrices are given by

$$\hat{\rho}_{\mathrm{out}}=\sum_q \tilde{p}_q\hat{P}_q,\quad \hat{\rho}_\alpha=\sum_{qij}\sqrt{p_ip_j}\langle j|\hat{\mathcal{E}}_q|i\rangle\hat{P}_q\otimes|\overline{i}\rangle\langle\overline{j}|, \tag{21}$$

where $\tilde{p}_q=\mathrm{Tr}\,\hat{\mathcal{E}}_q\hat{\rho}_{\mathrm{in}}$ are the state probabilities given by the indirect measurement.

In the case of full indirect measurement, it can be easily inferred theoretically or confirmed by numerical calculations for particular examples that no coherent information is available. The proof is based on the quantum analog [7] of the classical data processing theorem and the above discussed result on a full direct measurement. Therefore, in order to get nonzero coherent information, a class of incomplete (soft) measurements must be implemented, which are subject to more detailed quantum information analysis.

C. Quantum duplication procedure

In Sec. V B, we demonstrated that the classical-type measuring procedure defined by the transformation (17) completely destroys the coherent information transmitted through the quantum channel. Here we will consider a modified transformation for the quantum channel shown in Fig. 1(c), which preserves the coherent information,

$$\hat{\rho}_{12}\rightarrow\hat{\rho}'_{12}=\sum_{ij}\langle\phi_i|\mathrm{Tr}_2\hat{\rho}_{12}|\phi_j\rangle|\phi_i\rangle|\phi_i\rangle\langle\phi_j|\langle\phi_j|.$$

In this equation off-diagonal matrix elements of the input density matrix $\hat{\rho}_1=\hat{\rho}_{\mathrm{in}}$ are taken into account, which preserves the phase connections between different ϕ_i.

For the initial density matrix of a product type $\hat{\rho}_{\mathrm{in}}\otimes\hat{\rho}_2$, in terms of $\hat{\rho}_{\mathrm{in}}\rightarrow\hat{\rho}'_{12}$ transformation from H to $H\otimes H$, the corresponding superoperator has the form

$$\mathcal{Q}=\sum_{ij}|\phi_i\rangle|\phi_i\rangle\langle\phi_j|\langle\phi_j|\langle\phi_i|\odot|\phi_j\rangle. \tag{22}$$

This superoperator defines the coherent measuring transformation, in contrast to the incoherent transformation discussed in [31]. The coherent measuring transformation converts $\hat{\rho}_{\mathrm{in}}$ into a $\hat{\rho}_2$-independent state

$$\hat{\rho}_{\mathrm{out}}=\rho'_{12}=\sum_{ij}\langle\phi_i|\hat{\rho}_{\mathrm{in}}|\phi_j\rangle|\phi_i\rangle|\phi_i\rangle\langle\phi_j|\langle\phi_j|, \tag{23}$$

which results in the duplication of the input eigenstates ϕ_i into the same states of the pointer variable $\hat{k}=\Sigma_k k|\phi_k\rangle\langle\phi_k|$. Pure states of the input are transformed into the pure states of the joint $(1+2)$ system by doubling the pointer states

$$\psi \rightarrow \sum_i \langle \phi_i | \psi \rangle | \phi_i \rangle | \phi_i \rangle.$$

This mapping is similar to the mapping given by Eq. (16). Of course, only the input states ψ equal to the chosen pointer basis states ϕ_k are duplicated without distortion because it is impossible to transmit nonorthogonal states using only orthogonal ones. The entropy of the output state with a density matrix (23) having the same matrix elements as $\hat{\rho}_{\rm in}$ is evidently the same as the input state, $S_{\rm out}=S_{\rm in}=S[\hat{\rho}_{\rm in}]$, due to the preservation of the coherence of all pure input states.

For the joint input-output states, the transformation (22) yields the corresponding density matrix (6) in $H\otimes H\otimes H$ space,

$$\hat{\rho}_\alpha = \sum_{kl} |\phi_k\rangle|\phi_k\rangle\langle\phi_l|\langle\phi_l| \otimes \sqrt{\tilde{p}_k \tilde{p}_l}|\chi_k\rangle\langle\chi_l|, \tag{24}$$

where $\tilde{p}_k$, $|\chi_k\rangle$ are the same as above, providing an expansion of the input density matrix in the form $\hat{\rho}_{\rm in} = \Sigma_k \tilde{p}_k |\chi_k\rangle\langle\chi_k|$. Taking into account that the first tensor product term in Eq. (24) is a set of transition projectors $\hat{P}_{kl}, \hat{P}_{kl}\hat{P}_{mn}=\delta_{lm}\hat{P}_{kn}$, we can apply easily proven algebraic rules valid for a scalar function f:

$$f\left(\sum_{kl} \hat{P}_{kl}\otimes \hat{R}_{kl}\right) = \sum_{kl} \hat{P}_{kl}\otimes f(\hat{R})_{kl},$$

where $\hat{R}=(\hat{R}_{kl})$ is the block matrix and $\mathrm{Tr}\, f(\Sigma_{kl}\hat{P}_{kl}\otimes\hat{R}_{kl}) = \mathrm{Tr}\, f(\hat{R})$. Here $\hat{R}=(\sqrt{\tilde{p}_k\tilde{p}_l}|\chi_k\rangle\langle\chi_l|)$, and it is simply $||\chi\rangle\rangle\langle\langle\chi||^\dagger$ with $||\chi\rangle\rangle_{ki}=\sqrt{\tilde{p}_k}\chi_{ki}$, a vector in the $H\otimes H$ space. All eigenvalues λ_k of this matrix are equal to zero, except one value corresponding to the eigenvector $||\chi\rangle\rangle$.

Calculation of the exchange entropy gives $S_e=0$, and, therefore, $I_c=S_{\rm in}$. Consequently, the coherent duplication does not reduce the input information transmitted through the $1\rightarrow(1+2)$ channel, nor does it matter whether the register $\hat{k}$ is compatible with the input density matrix, $[\hat{k},\hat{\rho}_{\rm in}]=0$, or not.

If the channel is reduced to the one shown in Fig. 1(b) and discussed in Sec. V B, by taking in Eq. (23) trace either over the first or the second system, we evidently come to the measurement procedure discussed in Sec. V B. As a result, we can conclude that the coherent information is strictly associated with the joint system but not with its subsystems. This natural property could be used in quantum error correction algorithms [37] or for producing stable entangled states [38].

D. TLA-to-vacuum field channel

In this section we analyze the quantum channel between a TLA and a vacuum electromagnetic field [Fig. 1(b)], which is an extension of the TLA in an external laser field, as considered in Sec. III.

For this analysis, we will use a reduced model of the field, which is based on the reduction of the Hilbert space of the field in the Fock representation (Fig. 5). The problem, therefore, is reduced to that of the interaction of a two-level system with continuous multimode oscillator systems [39], a specific case of which is the interaction of an atom with the free photon field. However, to analyze the information in the system (atom+field), we do not need to consider the specific dependence of the wave function $\psi_0(\mathbf{k},\lambda)$ of the field photon on the wave vector (including polarization), because only its total probability and phase are significant.

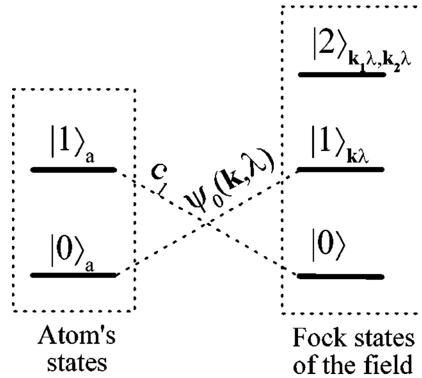

FIG. 5. Structure of the joint Hilbert space of the (atom+field) system. For the vacuum initial field state, both atomic states and only two Fock states of the field ($|0\rangle$ and $|1\rangle$) are involved in the dynamics of the joint system (atom+field). The dynamics is entirely defined by just two states, $|0\rangle_a|1\rangle_{\mathbf{k}\lambda}$ and $|1\rangle_a|0\rangle$, which are described by $\psi_0(\mathbf{k},\lambda)$ and c_1, respectively.

In the basis of the free atomic and field states for the vacuum's initial state $\alpha_0=0$, we get from Eq. (15)

$$S_{kl,\mu\nu} = \sum_m U_{m\mu,k0}U^*_{m\nu,l0}.$$

Greek letters are used to distinguish the photon field indexes, which in the general case include both the number of photons and their space or momentum coordinates. Matrix elements of this superoperator calculated via the atom-to-field unitary evolution matrix $U_{m\mu,k0}$ coefficients (Table I) are shown in Table II.

The choice of $\psi_0(\mathbf{k},\lambda)$ as a basis for the photon field [40] reduces the matrix of operator $S_{kl,\mu\nu}$ to the nonoperator matrix transformation, which in terms of $\hat{s}_{kl}$ matrices has the form

TABLE I. Unitary (atom+field) to (atom+field) transformation $U_{m\mu,k\alpha}$ for the vacuum initial photon-field state, where indexes m,k stand for atomic quanta and μ,α—for the number of photons. The long-dashed symbol stays for the elements not involved into the calculated terms $S_{kl,\mu\nu}$ (Table II).

$k\alpha$ \ $m\mu$	00	01	10	11
00	1	0	0	0
01	—	—	—	—
10	0	$\psi_0(\mathbf{k},\lambda)$	c_1	0
11	—	—	—	—

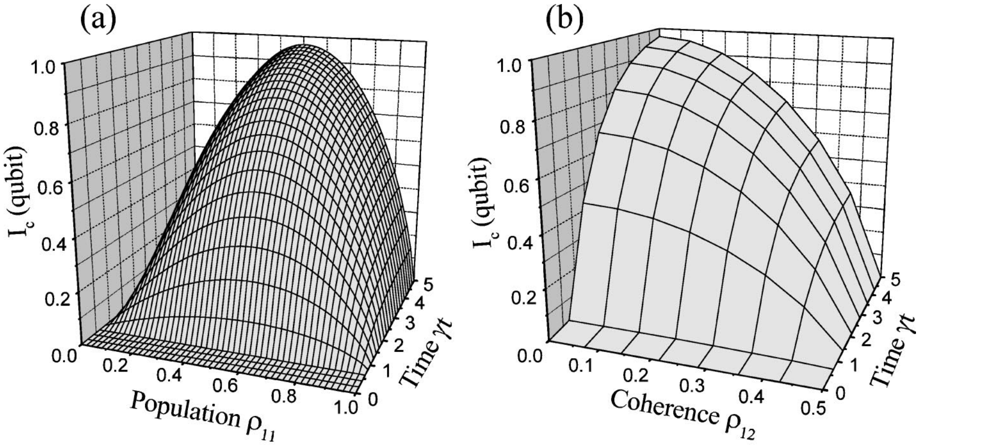

FIG. 6. The coherent information transmitted in the atom-to-field quantum channel versus the dimensionless time γt and input atomic density matrix, which is taken either diagonal with the ground-state matrix element ρ_{11} (a) or as the sum of $\hat{I}/2$ and the real (''cosine-type'') coherent contribution of the off-diagonal elements $\rho_{12}\hat{\sigma}_1$ (b).

$$\hat{s}_{11}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\quad \hat{s}_{12}=\begin{pmatrix}0 & (1-e^{-\gamma t})^{1/2}\\ 0 & 0\end{pmatrix},$$

$$\hat{s}_{21}=\begin{pmatrix}0 & 0\\ (1-e^{-\gamma t})^{1/2} & 0\end{pmatrix},\quad \hat{s}_{22}=\begin{pmatrix}e^{-\gamma t} & 0\\ 0 & 1-e^{-\gamma t}\end{pmatrix}, \tag{25}$$

where $|c_1|^2=\exp(-\gamma t)$ describes the population decay of the totally populated initial excited state of the atom and $\int\Sigma|\psi_0(\mathbf{k},\lambda)|^2 d\mathbf{k}=1-\exp(-\gamma t)$ is the probability a photon will be detected. From Eq. (25), it follows that the structure of the photon field plays no role, and the transmitted information defined by the input-output density matrix depends only on the photon emission probability by time t. The reduction of the photon field (only the photon numbers $\mu,\nu=0,1$ were taken into account) leads to the conclusion that the photon states also are equivalent to those of a two-level system.

Applying the transformation (25) to the input atom density matrix

$$\hat{\rho}_{\text{in}}=\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{12} & 1-\rho_{11}\end{pmatrix},$$

restricted to the real off-diagonal matrix elements, we get the output density matrix

$$\hat{\rho}_{\text{out}}=\begin{pmatrix}\rho_{11}+\rho_{22}e^{-\gamma t} & \rho_{12}(1-e^{-\gamma t})^{1/2}\\ \rho_{12}(1-e^{-\gamma t})^{1/2} & \rho_{22}(1-e^{-\gamma t})\end{pmatrix}$$

and for $\rho_{12}=0$ the respective input-output density matrix

$$\hat{\rho}_{\alpha}=\begin{pmatrix}\rho_{11} & 0 & 0 & [\rho_{11}\rho_{22}(1-e^{-\gamma t})]^{1/2}\\ 0 & \rho_{22}e^{-\gamma t} & 0 & 0\\ 0 & 0 & 0 & 0\\ [\rho_{11}\rho_{22}(1-e^{-\gamma t})]^{1/2} & 0 & 0 & \rho_{22}(1-e^{-\gamma t})\end{pmatrix}.$$

For $t\to\infty$ this expression yields a pure atom-photon state, which converts incoherent fluctuations of the atomic states, forming the incoherent ensemble, to equivalent coherent fluctuations of the photon states. The corresponding eigenvalues are $\lambda_{\alpha}=\{0,0,1-\rho_{22}\exp(-\gamma t),\rho_{22}\exp(-\gamma t)\}$. Nonzero values are equal to the probabilities of the atomic states at time t. For the output (photon) density matrix $\hat{\rho}_{\text{out}}$ the eigenvalues are $\lambda_{\text{out}}=\{\rho_{22}[1-\exp(-\gamma t)],1-\rho_{22}[1-\exp(-\gamma t)]\}$, which are the probability that a photon will be emitted or not. These sets of eigenvalues determine the eigenprobabilities of the joint input-output and marginal output matrices. The coherent information, defined by the difference of the corresponding entropies, then takes the form

$$I_c=x\rho_{22}\log_2(x\rho_{22})-(1-\rho_{22}+x\rho_{22})\log_2[1-(1-x)\rho_{22}]$$
$$+(1-x\rho_{22})\log_2(1-x\rho_{22})-(1-x)\rho_{22}\log_2(\rho_{22}-x\rho_{22}), \tag{26}$$

where $x=\exp(-\gamma t)$. This formula is valid for $I_c>0$, other-

TABLE II. Atom-to-field transformation $S_{kl,\mu\nu}$, which defines $|k\rangle\langle l|\to|\mu\rangle\langle\nu|$ superoperator transformation. Indexes k,l stand for atomic quanta and μ,ν—for the number of photons.

kl \ $\mu\nu$	00	01	10	11
00	1	0	0	0
01	0	0	$\psi_0(\mathbf{k},\lambda)$	0
10	0	$\psi_0^+(\mathbf{k},\lambda)$	0	0
11	$\|c_1\|^2$	0	0	$\psi_0(\mathbf{k},\lambda)\psi_0^+(\mathbf{k}',\lambda')$

wise, $I_c=0$. The corresponding critical point is $\exp(-\gamma t)=1/2$, the time when the probability $1-\rho_{22}[1-\exp(-\gamma t)]$ of finding no photon is equal to the population of the lower atomic state $1-\rho_{22}\exp(-\gamma t)$.

The results for calculating the coherent information are shown in Fig. 6 for two specific cases: $\rho_{12}=0$ [Fig. 6(a)] and $\rho_{11}=1/2$, $0\leqslant\rho_{12}\leqslant 1/2$ [Fig. 6(b)]. One can see from Fig. 6(a) that the coherent information is symmetrical with respect to the population ρ_{11} around the symmetry point $\rho_{11}=1/2$. Increasing the excited-state population $\rho_{22}=1-\rho_{11}$ and the corresponding photon emission yield does not increase the coherent information, because of the reduction of the source entropy, which determines the potential maximum value of the coherent information. For the same reason, the coherent information decreases when there is a nonzero coherent contribution to the initial maximum entropy atom state and completely vanishes for the pure coherent initial state [Fig. 6(b)].

In accordance with Sec. II and because of the purity of the initial field state, one-time information is equal to the difference of the entropies of the photon field only, represented by $\hat{\rho}_{\rm out}$, and the initial atomic state, represented by $\hat{\rho}_{\rm in}$. For a pure initial state, expressed in the form of the excited atom state $|2\rangle$, and for $0<t<\infty$, we always get nonzero information $I_c=-x\log_2 x-(1-x)\log_2(1-x)$ that yields 1 qubit for $x=1/2$, when the excited state population is equal to the probability a photon will be emitted.

E. The transmission of coherent information between two atoms via a free-space field

In this section we will consider the quantum channel when information is transmitted from one atom to another via the free-space field [Fig. 1(b)]. Suppose that the second atom is initially in the ground state. In addition, we will restrict ourselves here to the long time scale approximation, in which the effects of the discrete nature of the retarding electromagnetic interaction are neglected [41–44]. Under such restrictions and approximations we have the Dicke problem [45], for which the well-known solution for the atomic state in the form of two decaying symmetric and antisymmetric Dicke states $|s\rangle=(|1\rangle|2\rangle+|2\rangle|1\rangle)/\sqrt{2}$, $|a\rangle=(|1\rangle|2\rangle-|2\rangle|1\rangle)/\sqrt{2}$ and the stable vacuum state $|0\rangle=|1\rangle|1\rangle$ can be written as

$$c_s(t)=c_s(0)\exp[-(\gamma_s/2+i\Lambda)t],$$

$$c_a(t)=c_a(0)\exp[-(\gamma_a/2-i\Lambda)t],$$

$$c_0(t)=c_0(0)+[c_s(0)^2+c_a(0)^2-c_s(t)^2-c_a(t)^2]^{1/2}e^{i\xi(t)}. \tag{27}$$

Here $c_0(t)$ is the amplitude of the stable vacuum component $|1\rangle|1\rangle$, which has an incoherent contribution due to the spontaneous radiation transitions from the excited two-atomic states, $\xi(t)$ is the homogeneously distributed random phase, $\gamma_{s,a}$ and Λ are their decay rate and coupling shift, respectively, and $c_{s,a}$ are the amplitudes of the Dicke states.

In terms of the products of the individual atomic states $|i\rangle|j\rangle$ for the corresponding initial amplitudes $c_{12}(0)=0$, $c_{22}(0)=0$ the system's dynamics is described, according to the Dicke dynamics (27), by the following equations:

$$c_{11}(t)=c_{11}(0)+f(t)e^{i\xi(t)}c_{21}(0),\quad c_{21}(t)=f_s(t)c_{21}(0),$$

$$c_{12}(t)=f_a(t)c_{12}(0),\quad c_{22}(t)=0,$$

$$f(t)=\{1-[\exp(-\gamma_s t)+\exp(-\gamma_a t)]/2\}^{1/2},$$

$$f_s(t)=\{\exp[-(\gamma_s/2+i\Lambda)t]+\exp[-(\gamma_a/2-i\Lambda)t]\}/2,$$

$$f_a(t)=\{\exp[-(\gamma_s/2+i\Lambda)t]-\exp[-(\gamma_a/2-i\Lambda)t]\}/2.$$

Applying these formulas to the input operators $c_{k1}(0)c^*_{l1}(0)|k\rangle\langle l|$ of the first atom and then averaging the output over the final states of the first atom and the field fluctuations [the latter is represented here only with $\xi(t)$], we get the symbolic channel superoperator transformation $\hat{\rho}^{(1)}(0)\rightarrow\hat{\rho}^{(2)}(t)=\mathcal{S}(t)\hat{\rho}^{(1)}(0)$ and corresponding $\hat{s}_{kl}$ operators in the form

$$\begin{aligned}\mathcal{S}(t)=&|1\rangle\langle 1|\odot|1\rangle\langle 1|+[f(t)^2+|f_s(t)|^2]|1\rangle\langle 2|\odot|2\rangle\\&\times\langle 1|+|f_a(t)|^2|2\rangle\langle 2|\odot|2\rangle\langle 2|+f_a(t)|2\rangle\langle 2|\odot|1\rangle\\&\times\langle 1|+f_a^*(t)|1\rangle\langle 1|\odot|2\rangle\langle 2|,\end{aligned}$$

$$\hat{s}_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \hat{s}_{12}=\begin{pmatrix}0&f_a^*(t)\\0&0\end{pmatrix},$$

$$\hat{s}_{21}=\begin{pmatrix}0&0\\f_a(t)&0\end{pmatrix},\quad \hat{s}_{22}=\begin{pmatrix}f(t)^2+|f_s(t)|^2&0\\0&|f_a(t)|^2\end{pmatrix}. \tag{28}$$

To further elucidate this problem, let us now discuss the case of two identical atoms having parallel dipole moments aligned perpendicular to the vector connecting the atoms. Here only two-dimensionless parameters are essential: dimensionless time γt, where γ is the free atom's decay rate, and dimensionless distance, $\varphi=k_0R$, where R is the interatomic distance and k_0 is the wave vector at the atomic frequency. Then, the dimensionless two-atomic decay rates and the short distance dipole-dipole shift are given by [29,38,44]

$$\gamma_{s,a}/\gamma=1\pm g\quad\text{and}\quad\Lambda/\gamma=(3/4)/\varphi^3,$$

respectively, with $g=(3/2)(\varphi^{-1}\sin\varphi+\varphi^{-2}\cos\varphi-\varphi^{-3}\sin\varphi)$.

The coherent information may be calculated as previously described in Sec. V D by replacing $\exp(-\gamma t)$ with $f(t)^2+|f_s(t)|^2$ in Eq. (25). Then, the operators $\hat{s}_{kl}$ in Eq. (25) become similar to the corresponding operators in Eq. (28). The coherent information is given by the same Eq. (26) with $x=f(t)^2+|f_s(t)|^2$, which, however, now has (in contrast to a single-atom case considered in V D) new qualitative features arising from the specific oscillatory dependence of $|f_{s,a}(t)|^2$ on the interatomic distance φ.

If there were no oscillations from the quasielectrostatic dipole-dipole coupling, i.e., as in the case of $\Lambda=0$, the co-

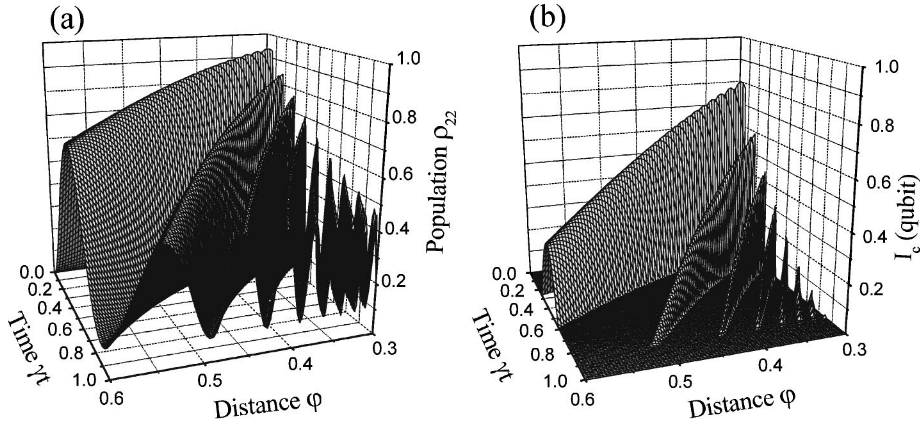

FIG. 7. Excited-state population of the second atom (a) and the coherent information (b) in a system of two atoms interacting via the free-space field versus time γt and the interatomic distance $\varphi = \omega_0 R/c$ (both are dimensionless). The input density matrix is diagonal with the ground-state matrix element $\rho_{22} = 1/2$.

herent information would always be equal to zero, because the threshold $x < 0.5$ would not be achieved. Parameter $(1 - x)$ corresponds to the population of the excited state of the second atom for the initial state $|2\rangle$ of the first atom, and for the optimal value $\rho_{22} = 1/2$ of its initial population (from the information point of view), we have $1 - x \leqslant 1/4$ and $x \geqslant 3/4$. Oscillations in $|f_a(t)|^2$ lead to the interference between the two decaying Dicke components, so that the maximum of the population $n_2 = 1 - x$ goes to the larger values, maximally up to $n_2 = 1$, and the coherent information becomes a nonzero value.

Functions $n_2(\varphi, \gamma t)$ and $I_c(\varphi, \gamma t)$, calculated with Eq. (26), are shown in Fig. 7. For the considered geometry, they serve as the universal measures for a system of two atoms independent of their frequency or dipole moments.

As can be seen from Fig. 7(a), the population decreases rapidly versus time because of the decay of the short-lived Dicke component. Both the population and the coherent information [Fig. 7(b)] show strong oscillations at smaller interatomic distances φ. At $\varphi \to 0$ the long-lived Dicke state yields an essential population even at infinitely long times, but it does not yield any coherent information after the total decay of the other short-lived Dicke state.

VI. CONCLUSIONS

In this paper we have shown that the coherent-information concept can be used effectively to quantify the interaction between two real quantum systems, which in the general case may have essentially different Hilbert spaces, and to elucidate the role of quantum coherence specific for the joint system.

For a TLA in a resonant laser field, coherent information in the system does not increase as the intensity of the external field increases, unless the external field modifies the relaxation parameters.

As an example of information transmission between the subsystems of a whole system, the hydrogen atom was considered. The coherent information in the atom was shown to transfer from the forbidden atomic transition to the dipole active transition in an external electric field, due to coupling through Stark splitting.

For two unitary coupled TLAs, the maximum value $I_c = 1$ qubit of the coherent information was shown to be achieved for a complete unitary entanglement of two TLAs and $I_c = 0$, for any kind of measuring procedure discussed in Sec. V B.

For the information exchange between a TLA and a free-space vacuum photon field via spontaneous emission, the coherent information was shown to reach a nonzero value at the threshold point of the decay exponent $\exp(-\gamma t)$ equal to 1/2, when the probability of finding no photon is equal to the population of the lower atomic state. At its maximum, the coherent information can reach the value of $I_c = 1$ qubit.

For the information transfer between two atoms via vacuum field, when the atoms are located at a distance of the order of their transition wavelength, the coherent information was shown to be a nonzero value, only because of the coherent oscillations of the Dicke states, which originate from the dipole-to-dipole short distance electrostaticlike $\sim 1/R^3$ interaction. In contrast, the semiclassical information received from the quantum detection procedure results from the population correlations [38].

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